

A Characterisation of Locality in Momentum Space

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Abstract

It is proved that a Poincaré invariant Wightman function which fulfils the spectral property and can be defined at sharp times is local if and only if the integration over both the energy variables of a commutator in momentum space is a polynomial in the momentum conjugated to the spacial difference variable of the commutator with distributional coefficients depending on the remaining energy and momentum variables. Using this characterisation of locality in momentum space, the locality of a sequence of Wightman functions with nontrivial scattering behaviour (associated to some quantum field in indefinite metric) can be proved by explicit calculations. We compare the above characterisation of locality with the classical integral representation method of Jost, Lehmann and Dyson.

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Introduction

The vacuum expectation values (Wightman functions) of a local relativistic quantum field contain all the general and specific information of the under-

lying quantum field theory (QFT).

The formulation of the general (axiomatic) properties of (truncated) Wightman functions given in the classical literature [8, 11] requires expressions for the Wightman functions in position space and in momentum space. In applications however, the passage from one picture to the “dual picture” by the Fourier transform requires quite nontrivial calculations. It would therefore be convenient to get formulations for all axiomatic properties in only one of these “pictures”.

It seems to be somehow more natural to study (truncated) Wightman functions in momentum space than in position space, since in momentum space there are direct formulations for the spectral property and cluster property in terms of the support of the (truncated) Wightman functions. Furthermore, Poincaré invariance, mass shell singularities leading to nontrivial scattering behaviour etc. can be directly identified from (truncated) Wightman functions in momentum space. It therefore is an interesting problem to get a description of locality in momentum space.

A characterisation of locality in momentum space has been given by Jost, Lehmann and Dyson (JLD) in the form of an integral representation of causal commutators [6, 9]. While this integral representation is a very powerful tool to investigate structural properties (e.g. analyticity) of causal commutators in momentum space, it seems non-trivial to decide the question of the locality of a given Wightman function in momentum space on the basis of the JLD–representation, since this amounts more or less to the calculation of the inverse of an integral transform. Also, as we shall demonstrate in this article, the JLD–representation tacitly requires some regularity assumptions, which rule out some cases of commutators of local Wightman functions.

Here we give a new characterisation of locality in momentum space which seems to give a very straight forward criterium to check locality in momentum space. This is being illustrated by a concrete application to a physically nontrivial situation. We also apply our method to the JLD–representation and we provide examples which show that our characterisation method for causal commutators goes properly beyond the result of [6, 9].

The article is organised as follows: After collecting some notations and definitions in Section 1 we propose such a criterion for locality in momentum space (Section 2). We then in Section 3 apply this criterion to a sequence of truncated Wightman functions (associated to a physically nontrivial QFT in “indefinite metric”, cf. [1, 3, 4, 10]) which were constructed in [2] (see also the references in this article). The proof of locality obtained along this line

is much shorter and simpler than the proof of the same statement based on Euclidean QFT and analytic continuation which is given in [2]. Finally, in Section 4 we compare our method with the integral representation method of JLD [6, 9].

1 Double integrals over the energy variables of a commutator

Let \mathbb{R}^d , $d \geq 2$, be the d -dimensional Minkowski space time with inner product $x \cdot y = x^0 y^0 - \vec{x} \cdot \vec{y}$, $x = (x^0, \vec{x})$, $y = (y^0, \vec{y}) \in \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^d$. For $x \cdot x$ we also use the expression x^2 . We denote the forward (backward) light cone $\{x \in \mathbb{R}^d : x^2 > 0, x^0 > 0 (x^0 < 0)\}$ by $V_0^+(V_0^-)$ and $\bar{V}_0^+(\bar{V}_0^-)$ is the closed forward (backward) light cone.

We deal with the n -point vacuum expectation values of a QFT with N species of quantum fields labeled by indices κ_l , $l = 1, \dots, n$, which transform covariantly under spin representations T_{κ_l} of the covering group of the Lorentz group \tilde{L}_+^\uparrow over \mathbb{R}^d with finite dimensional spin space E_{κ_l} . The index $\nu_l = 1, \dots, \dim_{\mathbb{C}} E_{\kappa_l}$ is the spin-index of the quantum field of species κ_l .

By \mathcal{S}_n we denote the space of Schwartz testfunctions over \mathbb{R}^{dn} with values in $E^{\otimes n}$ where $E = \bigoplus_{\kappa=1}^N E_{\kappa}$. A (truncated) n -point Wightman function W_n is an element in the topological dual space \mathcal{S}'_n of \mathcal{S}_n .

The components of the Wightman functions are denoted by $W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n)$ and their Fourier transform is defined as

$$\begin{aligned} \hat{W}_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(k_1, \dots, k_n) &= (2\pi)^{-nd/2} \int_{\mathbb{R}^{dn}} e^{-i(k_1 \cdot x_1 + \dots + k_n \cdot x_n)} \\ &\times W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned} \quad (1)$$

Here the integral on the right hand side has to be understood in the sense of the Fourier transform of tempered distributions, cf. [5]. In the following, we also use the symbol \mathcal{F} for the Fourier transform and we denote the inverse Fourier transform by $\bar{\mathcal{F}}$.

We assume that W_n fulfils the spectral property $\text{supp } \hat{W}_n \subseteq \{(k_1, \dots, k_n) \in \mathbb{R}^{dn} : \sum_{l=j}^n k_l \in \bar{V}_0^+, j = 1, \dots, n-1\}$ and the property of

Poincaré invariance

$$\begin{aligned} & \prod_{l=1}^n T_{\kappa_l, \nu_l}^{\nu'_l}(\Lambda) W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(\Lambda^{-1}(x_1 - a), \dots, \Lambda^{-1}(x_n - a)) \\ &= W_n^{(\kappa_1, \dots, \kappa_n) \nu'_1 \dots \nu'_n}(x_1, \dots, x_n) \quad \forall \Lambda \in \tilde{L}_+^\uparrow, \quad a \in \mathbb{R}^d, \end{aligned} \quad (2)$$

where we applied the Einstein convention of summation (ECS), i.e. any spin index ν_l is summed up over $1, \dots, \dim_{\mathbb{C}} E_{\kappa_l}$.

The desired (anti-) commutation relations of a field of type κ with a field of type κ' are being fixed by a symmetric $N \times N$ -matrix σ , $\sigma^{\kappa, \kappa'} = \pm 1$, $\kappa, \kappa' = 1, \dots, N$.

By definition, a (truncated) Wightman function W_n is local (w.r.t. σ) if and only if for $x_j - x_{j+1}$ is space-like (i.e. $(x_j - x_{j+1})^2 < 0$) we get

$$\begin{aligned} & W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) = \\ & \sigma^{\kappa_j, \kappa_{j+1}} W_n^{(\kappa_1, \dots, \kappa_{j+1}, \kappa_j, \dots, \kappa_n) \nu_1 \dots \nu_{j+1} \nu_j \dots \nu_n}(x_1, \dots, x_{j+1}, x_j, \dots, x_n). \end{aligned} \quad (3)$$

For $j = 1, \dots, n-1$, we define the distribution $W_{n, [j]} \in \mathcal{S}'_n$ by

$$\begin{aligned} & W_{n, [j]}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) \\ &= W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, [x_j, x_{j+1}], \dots, x_n) \\ &= W_n^{(\kappa_1, \dots, \kappa_j, \kappa_{j+1}, \dots, \kappa_n) \nu_1 \dots \nu_j \nu_{j+1} \dots \nu_n}(x_1, \dots, x_j, x_{j+1}, \dots, x_n) \\ &- \sigma^{\kappa_j, \kappa_{j+1}} W_n^{(\kappa_1, \dots, \kappa_{j+1}, \kappa_j, \dots, \kappa_n) \nu_1 \dots \nu_{j+1} \nu_j \dots \nu_n}(x_1, \dots, x_{j+1}, x_j, \dots, x_n) \end{aligned} \quad (4)$$

Let φ be a symmetric, real Schwartz function on \mathbb{R} with support in $[-1, 1]$ and $\int_{\mathbb{R}} \varphi dx = 1$. We set $\varphi_\epsilon(x) = \varphi(x/\epsilon)/\epsilon$. We define $W_{n, j, \epsilon}$ as the distribution W_n convoluted with φ_ϵ in each of the arguments x_j^0, x_{j+1}^0 . The distribution $W_{n, j, \epsilon}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n)$ thus is a smooth and polynomially bounded function in the arguments x_j^0, x_{j+1}^0 , provided it is smeared out with some testfunction in the remaining arguments $x_1, \dots, x_{j-1}, \vec{x}_j, \vec{x}_{j+1}, x_{j+2}, \dots, x_n$. Furthermore, $\lim_{\epsilon \rightarrow +0} W_{n, j, \epsilon} = W_n$ for $j = 1, \dots, n-1$. For $W_{n, [j], j, \epsilon}$ we write $W_{n, [j], \epsilon}$.

Frequently we need the testfunction spaces $\mathcal{S}_{n, j}$, $j = 1, \dots, n-1$ which are the spaces of Schwartz functions of the arguments $x_1, \dots, x_{j-1}, \vec{x}_j, \vec{x}_{j+1}, x_{j+2}, \dots, x_n$ with values in $E^{\otimes n}$ with the Schwartz topology

and their topological dual spaces $\mathcal{S}'_{n,j}$. By $W_{n,j,\epsilon}(s, t) \in \mathcal{S}'_{n,j}$ we denote the distribution which is defined by

$$\begin{aligned} W_{n,j,\epsilon}(s, t)(f) &= \int_{\mathbb{R}^{dn}} W_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) \\ &\times f_{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_{j-1}, \vec{x}_j, \vec{x}_{j+1}, x_{j+2}, \dots, x_n) \\ &\times \varphi_\epsilon(s - x_j^0) \varphi_\epsilon(t - x_{j+1}^0) dx_1 \dots dx_n \end{aligned} \quad (5)$$

where we applied the ECS to the indices $\kappa_l, \nu_l, l = 1, \dots, n$.

In order to formulate our condition of locality, we need the following rather weak and technical restriction on the Wightman functions W_n :

Condition 1.1 We say that W_n fulfils the weak time zero field condition, if $W_{n,j}(0, 0) = \lim_{\epsilon \rightarrow +0} W_{n,j,\epsilon}(0, 0)$ exists in $\mathcal{S}'_{n,j}$ for $j = 1, \dots, n-1$.

Remark 1.2 (i) Condition 1.1 follows from the existence of sharp time fields $\phi(\delta_t \otimes f)$ where $f \in \mathcal{S}_1(\mathbb{R}^{d-1}, E)$ and $\delta_t(x^0) = \delta(x^0 - t)$ at time zero. However, no precise assumptions are made on the domains of definition of such quantum fields, and we therefore labeled it with the adjective "weak".

(ii) Of course, if the weak time zero field condition holds for W_n , then it also holds for $W_{n,[j]}$ and we get a distribution $W_{n,[j]}(0, 0)$ in $\mathcal{S}'_{n,j}$.

(iii) Formally we get the following expression for the Fourier transform (in $\mathcal{S}'_{n,j}$) $\hat{W}_{n,[j]}(0, 0)$ of $W_{n,[j]}(0, 0)$:

$$\begin{aligned} &\hat{W}_{n,[j]}(0, 0)(k_1, \dots, k_{j-1}, \vec{k}_j, \vec{k}_{j+1}, k_{j+1}, \dots, k_n) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{W}_n(k_1, \dots, [k_j, k_{j+1}], \dots, k_n) dk_j^0 dk_{j+1}^0 \end{aligned} \quad (6)$$

For $j = 1, \dots, n-1$ we define $\xi_+ = (x_j + x_{j+1})/2$ and $\xi_- = (x_j - x_{j+1})/2$. The variables conjugated to $x_1, \dots, x_{j-1}, \xi_-, \xi_+, x_{j+1}, \dots, x_n$ under the Fourier transform are $k_1, \dots, k_{j-1}, q_-, q_+, k_{j+1}, \dots, k_n$ with $q_\pm = (k_j \pm k_{j+1})/2$. We define another testfunction space $\mathcal{S}_{n,j,+}$ as the space of Schwartz functions on $\mathbb{R}^{d(j-1)} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d(n-j-2)}$ with values in $E^{\otimes n}$. Given $f \in \mathcal{S}_{n,j,+}$ we define the tempered distribution in the argument \vec{q}_- , $\hat{W}_{n,[j]}(0, 0)(f)(\vec{q}_-)$, as

$$\begin{aligned} \hat{W}_{n,[j]}(0, 0)(f)(\vec{q}_-) &= \int_{\mathbb{R}^{d(n-2)+(d-1)}} \hat{W}_{n,[j]}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(0, 0)(k_1, \dots, \\ &k_{j-1}, \vec{q}_+ + \vec{q}_-, \vec{q}_+ - \vec{q}_-, k_{j+2}, \dots, k_n) \\ &\times f_{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(k_1, \dots, k_{j-1}, \vec{q}_+, k_{j+2}, \dots, k_n) \\ &\times dk_1 \dots dk_{j-1} d\vec{q}_+ dk_{j+2} \dots dk_n, \end{aligned} \quad (7)$$

where we again used the ECS. By an analogous formula we define $W_{n,[,]j}(0,0)(f)(\vec{\xi}_-)$ for $f \in \mathcal{S}_{n,j,+}$. If $f \in \mathcal{S}_{n,j}$ we then define $f_+(\vec{\xi}_-) \in \mathcal{S}_{n,j,+}$ for $\vec{\xi}_- \in \mathbb{R}^{d-1}, j = 1, \dots, n-1$ as

$$\begin{aligned} & f_+(\vec{\xi}_-)(x_1, \dots, x_{j-1}, \vec{\xi}_+, x_{j+2}, \dots, x_n) \\ &= f(x_1, \dots, x_{j-1}, \vec{\xi}_+ + \vec{\xi}_-, \vec{\xi}_+ - \vec{\xi}_-, x_{j+2}, \dots, x_n). \end{aligned} \quad (8)$$

Then, we get

$$W_{n,[,]j}(0,0)(f) = \int_{\mathbb{R}^{d-1}} W_{n,[,]j}(0,0)(f_+(\vec{\xi}_-))(\vec{\xi}_-) d\vec{\xi}_- \quad \forall f \in \mathcal{S}_{n,j}. \quad (9)$$

where the integral is a symbolic ("distributional") integral.

2 The main theorem

We have now finished the preparations for the formulation of the following criterion for locality in momentum space:

Theorem 2.1 *Let $W_n \in \mathcal{S}'_n$ be a Poincaré invariant distribution which fulfils the spectral property and the weak time zero field condition 1.1. Then W_n is local if and only if $\hat{W}_{n,[,]j}(0,0)(f)(\vec{q}_-)$ is a polynomial in \vec{q}_- for all $f \in \mathcal{S}_{n,j,+}$.*

Proof. \Rightarrow : Let $j \in \{1, \dots, n-1\}$ and $f \in \mathcal{S}_{n,j,+}$ be fixed.

We note that the polynomials are the Fourier transforms of the complex linear combinations of the delta distribution in the point $0 \in \mathbb{R}^{(d-1)}$ and it's partial derivatives.

Thus, the distribution $\hat{W}_{n,[,]j}(0,0)(f)(\vec{q}_-)$ is a polynomial in \vec{q}_- if and only if its inverse Fourier transform $\bar{\mathcal{F}}_{\vec{q}_-}(\hat{W}_{n,[,]j}(0,0)(f))(\vec{\xi}_-)$ is a linear combination of the delta distribution in the point $0 \in \mathbb{R}^{(d-1)}$ and its partial derivatives.

By [5] p. 56 we know that the distributions with support in $\{0\} \subseteq \mathbb{R}^d$ are just given by the linear combination of the delta distribution in the point $0 \in \mathbb{R}^{(d-1)}$ and its partial derivatives. It is thus equivalent to show that $\text{supp } \bar{\mathcal{F}}_{\vec{q}_-}(\hat{W}_{n,[,]j}(0,0)(f))(\vec{\xi}_-) \subseteq \{0\}$. Let $B_{\epsilon_1}(0) \subseteq \mathbb{R}^{d-1}$ be the ball of radius ϵ_1 with center 0. We have to show that for any $\epsilon_1 > 0$ and Schwartz function $h(\vec{\xi}_-)$ with support in $\mathbb{R}^{d-1} - B_{\epsilon_1}(0)$ we have

$$\int_{\mathbb{R}^{d-1}} \bar{\mathcal{F}}_{\vec{q}_-}(\hat{W}_{n,[,]j}(0,0)(f))(\vec{\xi}_-) h(\vec{\xi}_-) d\vec{\xi}_- = 0.$$

Rewriting this equation in terms of $W_{n,[,]_j}$ we get for the left hand side

$$\begin{aligned} & \lim_{\epsilon \rightarrow +0} \int_{\mathbb{R}^{dn}} W_{n,[,]_j}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) \\ & (\bar{\mathcal{F}}f)_{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_{j-1}, \vec{x}_j + \vec{x}_{j+1}, x_{j+2}, \dots, x_n) \\ & \times h((\vec{x}_j - \vec{x}_{j+1})/2) \varphi_\epsilon(x_j^0) \varphi_\epsilon(x_{j+1}^0) dx_1 \dots dx_n. \quad (ECS) \end{aligned}$$

We note that for $0 < 4\epsilon < \epsilon_1$ the support of the function $h((\vec{x}_j - \vec{x}_{j+1})/2) \times \varphi_\epsilon(x_j^0) \varphi_\epsilon(x_{j+1}^0)$ is contained in $\{(x_j, x_{j+1}) \in \mathbb{R}^d \times \mathbb{R}^d : (x_j - x_{j+1})^2 < 0\}$. Thus, by the locality of W_n , the "integral" in the above expression is zero for such ϵ and thus the limit is zero.

\Leftarrow : First we fix some notations and recall some results of axiomatic QFT following [11].

By the spectral property, the Wightman functions $W_n^{(\kappa_1, \dots, \kappa_n) \nu_1, \dots, \nu_n}(x_1, \dots, x_n)$ are boundary values of holomorphic functions $\mathcal{W}_n^{(\kappa_1, \dots, \kappa_n) \nu_1, \dots, \nu_n}(z_1, \dots, z_n)$, $z_j = x_j + iy_j$, $j = 1, \dots, n$ which are analytic in the tube $\mathcal{T}_n = \mathbb{R}^n + i\Gamma_n$ with $\Gamma_n = \{(y_1, \dots, y_n) \in \mathbb{R}^{dn} : y_j - y_{j+1} \in V_0^-\}$. Let $\tilde{L}_+(\mathbb{C}^d)$ be the (covering group of the) proper complex Lorentz group. Then, by Poincaré invariance, $\mathcal{W}_n^{(\kappa_1, \dots, \kappa_n) \nu_1, \dots, \nu_n}(z_1, \dots, z_n)$ has a single valued extension to the extended tube $\mathcal{T}'_n = \tilde{L}_+(\mathbb{C}^d) \cdot \mathcal{T}_n$ (here the dot stands for the diagonal action of $\tilde{L}_+(\mathbb{C}^d)$ on \mathbb{C}^{dn}). The real points in \mathcal{T}'_n are the so called Jost points, i.e. the points $\{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : (\sum_{j=1}^{n-1} \lambda_j (x_j - x_{j+1}))^2 < 0 \forall \lambda_j \geq 0, \sum_{j=1}^{n-1} \lambda_j > 0\}$.

Similarly, for $\pi \in \text{Perm}(n)$ the permuted Wightman function $W_{n,\pi}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(x_1, \dots, x_n) = W_n^{(\kappa_{\pi_1}, \dots, \kappa_{\pi_n}) \nu_{\pi_1}, \dots, \nu_{\pi_n}}(x_{\pi_1}, \dots, x_{\pi_n})$ is the boundary value of a holomorphic function $\mathcal{W}_{n,\pi}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(z_1, \dots, z_n)$ defined on the extended tube $\mathcal{T}'_{n,\pi} = p(\pi) \cdot \mathcal{T}'_n$ where p denotes the action of the permutation group on \mathbb{C}^{dn} .

For $j = 1, \dots, n-1$ let $(j, j+1) \in \text{Perm}(n)$ denote the transposition of j and $j+1$. Then, \mathcal{T}'_n and $\mathcal{T}'_{n,(j,j+1)}$ have a nonempty real open intersection $\mathcal{N}_{n,(j,j+1)}$. Furthermore, since the action of \tilde{L}_+^\uparrow maps real points to real points and leaves \mathcal{T}'_n and $\mathcal{T}'_{n,(j,j+1)}$ invariant, $\mathcal{N}_{n,(j,j+1)}$ is invariant under the action of \tilde{L}_+^\uparrow . Similarly, $\mathcal{N}_{n,(j,j+1)}$ is invariant under (real) translations. If

$$\mathcal{W}_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(z_1, \dots, z_n) = \sigma^{\kappa_j, \kappa_{j+1}} \mathcal{W}_{n,(j,j+1)}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(z_1, \dots, z_n) \quad (10)$$

holds on $\mathcal{N}_{n,(j,j+1)}$, then $\mathcal{W}_n^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(z_1, \dots, z_n)$ and $\mathcal{W}_{n,(j,j+1)}^{(\kappa_1, \dots, \kappa_n) \nu_1 \dots \nu_n}(z_1, \dots, z_n)$ have single valued continuation on $\mathcal{T}'_n \cup \mathcal{T}'_{n,(j,j+1)}$

and the relation (10) holds on this domain.

Since the transpositions generate the group of permutations, it is sufficient to prove (10) for $j = 1, \dots, n-1$ in order to obtain analytic functions $\mathcal{W}_{n,\pi}^{(\kappa_1, \dots, \kappa_n)\nu_1, \dots, \nu_n}(z_1, \dots, z_n)$ defined on the permuted extended tube $\mathcal{T}_n^{p.e.} = \cup_{\pi \in \text{Perm}(n)} \mathcal{T}'_{\pi,n}$, s.t. the relation (10) and related relations between these functions hold on $\mathcal{T}_n^{p.e.}$. If this is true, then, by a general theorem of R. Jost [8] p. 83, W_n is local.

It is thus sufficient to prove Equation (10) on $\mathcal{N}_{n,(j,j+1)}$ for $j = 1, \dots, n-1$. Since the points in $\mathcal{N}_{n,(j,j+1)}$ are real, this equation can be written in terms of the Wightman functions themselves, i.e. we have to show that

$$\begin{aligned} & W_n^{(\kappa_1, \dots, \kappa_n)\nu_1 \dots \nu_n}(x_1, \dots, x_n) \\ &= \sigma^{\kappa_j, \kappa_{j+1}} W_n^{(\kappa_1, \dots, \kappa_{j+1}, \kappa_j, \dots, \kappa_n)\nu_1 \dots \nu_{j+1} \nu_j \dots \nu_n}(x_1, \dots, x_{j+1}, x_j, \dots, x_n) \end{aligned} \quad (11)$$

holds for $(x_1, \dots, x_n) \in \mathcal{N}_{n,(j,j+1)}$. We note that the above equation is a relation between real analytic functions and therefore no smearing in the variables x_1, \dots, x_n is required in order to make it rigorous. We can thus fix (x_1, \dots, x_n) .

Since $(x_1, \dots, x_n) \in \mathcal{N}_{n,(j,j+1)}$, (x_1, \dots, x_n) is a Jost point and we get that $x_j - x_{j+1}$ is space like. Thus there exists a Lorentz transformation $\Lambda \in L_+^\uparrow$ s.t. Λ^{-1} maps $x_j - x_{j+1}$ to the hyperplane $\{0\} \times \mathbb{R}^{d-1}$. Equation (11) thus is equivalent with

$$\begin{aligned} & \prod_{l=1}^n T_{\kappa_l, \nu_l}^{\nu'_l}(\Lambda) W_n^{(\kappa_1, \dots, \kappa_n)\nu_1 \dots \nu_n}(\Lambda^{-1}x_1, \dots, \Lambda^{-1}x_n) \\ &= \sigma^{\kappa_j, \kappa_{j+1}} \prod_{l=1}^n T_{\kappa_l, \nu_l}^{\nu'_l}(\Lambda) W_n^{(\kappa_1, \dots, \kappa_{j+1}, \kappa_j, \dots, \kappa_n)\nu_1 \dots \nu_{j+1} \nu_j \dots \nu_n}(\Lambda^{-1}x_1, \\ & \dots, \Lambda^{-1}x_{j+1}, \Lambda^{-1}x_j, \dots, \Lambda^{-1}x_n) \end{aligned}$$

where we applied the ECS to the indices ν_l . It is therefore sufficient to prove Equation (11) for the points $(x_1, \dots, x_n) \in \mathcal{N}_{n,(j,j+1)}$ replaced by the points $(x'_1, \dots, x'_n) = (\Lambda^{-1}x_1, \dots, \Lambda^{-1}x_n) \in \mathcal{N}_{n,(j,j+1)}$, where $x'_j = x'^0_{j+1}$. Furthermore, by the translation invariance of the Wightman functions and the translation invariance on $\mathcal{N}_{n,(j,j+1)}$, this is equivalent with Equation (11) for the points (x_1, \dots, x_n) replaced with the points $(x''_1, \dots, x''_n) \in \mathcal{N}_{n,(j,j+1)}$ where $x''_l = (x'^0_l - x'^0_j, \vec{x}'_l)$, $l = 1, \dots, n$. It is thus sufficient to prove Equation

(11) for points $(x_1, \dots, x_n) \in \mathcal{N}_{n,(j,j+1)}$ with $x_j^0 = x_{j+1}^0 = 0$. We denote the set of these points by $\mathcal{N}_{n,(j,j+1)}^0$. Since $\mathcal{N}_{n,(j,j+1)}$ is open in \mathbb{R}^{dn} , we can also consider $\mathcal{N}_{n,(j,j+1)}^0$ as an open subset of $\mathbb{R}^{d(j-1)} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d(n-j-2)}$. That Equation (11) holds on $\mathcal{N}_{n,(j,j+1)}^0$ thus is equivalent to

$$W_{n,[,]_j}(0,0)(f) = 0 \quad \forall f \in \mathcal{S}_{n,j}, \quad \text{supp } f \subseteq \mathcal{N}_{n,(j,j+1)}^0.$$

By equation (9) it is sufficient to show that the distribution (in $\vec{\xi}_-$) $W_{n,[,]_j}(0,0)(f_+(\vec{\xi}_-))(\vec{\xi}_-) = 0$ for $f \in \mathcal{S}_{n,j}$ with $\text{supp } f \subseteq \mathcal{N}_{n,(j,j+1)}^0$. As in the first part of the proof we get from the fact that $\hat{W}_{n,[,]_j}(0,0)(\mathcal{F}f_+(\vec{\xi}_-))(\vec{q}_-)$ is a polynomial in \vec{q}_- (here the Fourier transform \mathcal{F} is the Fourier transform in $\mathcal{S}_{n,j,+}$ and $\vec{\xi}_-$ is a fixed parameter) that $W_{n,[,]_j}(0,0)(f(\vec{\xi}_-))(\vec{\xi}_-)$ has support concentrated in $\{0\} \subseteq \mathbb{R}^{d-1}$. We note that for $(x_1, \dots, x_{j-1}, \vec{x}_j, \vec{x}_{j+1}, x_{j+2}, \dots, x_n) \in \mathcal{N}_{n,(j,j+1)}^0$ we have $\vec{x}_j \neq \vec{x}_{j+1}$, since $\mathcal{N}_{n,(j,j+1)}^0$ consists of Jost points which implies $(0, \vec{x}_j - \vec{x}_{j+1})^2 < 0$. Thus, $f_+(\vec{\xi}_-) = 0$ on a neighbourhood of $0 \in \mathbb{R}^{d-1}$. Consequently, $W_{n,[,]_j}(0,0)(f_+(\vec{\xi}_-))(\vec{\xi}_-) = 0$ holds on this neighbourhood and therefore holds everywhere. ■

3 Application: Locality of the structure functions

As an immediate consequence of Theorem 2.1, one obtains the locality of the two point function $\hat{W}_2(k_1, k_2) = \delta_m^-(k_1)\delta(k_1 + k_2)$ of the free field of mass m as follows (here $\delta_m^\pm(k) = \theta(\pm k^0)\delta(k^2 - m^2)$ with θ beeing the Heaviside function): It is well-known, that spectrality and Poincaré invariance hold for this distribution and that also the weak time zero field condition holds. Thus, the following short calculation suffices to prove locality:

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{W}_{2,[,]_1}(k_1, k_2) dk_1^0 dk_2^0 &= \int_{\mathbb{R}} \int_{\mathbb{R}} (\delta_m^-(k_1) - \delta_m^-(k_2)) \\ &\times \delta(k_1 + k_2) dk_1^0 dk_2^0 = \left(\frac{1}{2\omega_1} - \frac{1}{2\omega_2} \right) \delta(\vec{k}_1 + \vec{k}_2) = 0 \end{aligned} \quad (12)$$

where $\omega_j = \sqrt{|\vec{k}_j|^2 + m^2}$, $j = 1, 2$.

But in this section we want to show that Theorem 2.1 is useful especially in physically nontrivial situations. To do this, we define a sequence of

truncated Wightman functions, called the structure functions, which play a crucial rôle in the construction of quantum fields in indefinite metric with nontrivial scattering behaviour given in [1, 2, 3, 4, 10]. In fact, the scattering amplitudes associated to the structure functions just consist of “on shell” and energy-momentum conservation terms. Using the characterisation of locality in momentum space, we derive the locality of these truncated Wightman functions. This result is implicitly already contained in [2]. However, the proof given there uses “Euclidean” methods and analytic continuation and is much longer than the proof we present here.

For $n \in \mathbb{N}, n \geq 3$, let $m_1, \dots, m_N \in \mathbb{R}^n, m_\kappa > 0$ if $d = 2, 3$ and $m_\kappa \geq 0$ for $d \geq 4, \kappa = 1, \dots, N$. Let $\bar{\kappa} = (\kappa_1, \dots, \kappa_n)$. We then define the distributions

$$\hat{G}_{n,\bar{\kappa}}(k_1, \dots, k_n) = \left\{ \sum_{j=1}^n \prod_{l=1}^{j-1} \delta_{m_{\kappa_l}}^-(k_l) \frac{1}{k_j^2 - m_{\kappa_j}^2} \prod_{l=j+1}^n \delta_{m_{\kappa_l}}^+(k_l) \right\} \delta\left(\sum_{l=1}^n k_l\right) \quad (13)$$

Here the singularities $1/(k_j^2 - m_{\kappa_j}^2)$ have to be understood in the sense of Cauchy’s priciple value, cf. [5] p.44.

Definition 3.1 For $n \geq 3$ we define the structure function G_n as the inverse Fourier transform of the distribution \hat{G}_n given by $\hat{G}_n = \sum_{\kappa_1, \dots, \kappa_n=1}^N \hat{G}_{n,\bar{\kappa}}$.

The distributions \hat{G}_n are manifestly Poincaré invariant. Furthermore, they fulfil the spectral condition, which can be proved as follows: Let (k_1, \dots, k_n) be in the support of the j -th summand of $\hat{G}_{n,\bar{\kappa}}$. Then $\sum_{l=r}^n k_l \in \bar{V}_0^+$ for $r = j+1, \dots, n-1$ since $k_l \in \bar{V}_0^+$ for $l = r, \dots, n-1$. If $1 \leq r \leq j$ then $\sum_{l=r}^n k_l = -\sum_{l=1}^{r-1} k_l \in \bar{V}_0^+$ since $k_l \in -\bar{V}_0^+$ for $l = 1, \dots, j-1$. In order to apply Theorem 2.1, it remains to prove the following lemma:

Lemma 3.2 For $n \geq 3$ the structure function G_n fulfils the weak time zero field condition 1.1.

Proof. By Parseval’s theorem, the weak sharp time field condition for G_n is equivalent to the existence (in $\mathcal{S}'_{n,j}$) of the limit

$$\begin{aligned} & \lim_{\epsilon \rightarrow +0} \int_{\mathbb{R}^{dn}} \hat{G}_n(k_1, \dots, k_n) \hat{\varphi}_\epsilon(k_j^0) \hat{\varphi}_\epsilon(k_{j+1}^0) \\ & \times f(k_1, \dots, k_{j-1}, \vec{k}_j, \vec{k}_{j+1}, k_{j+2}, \dots, k_n) dk_1 \cdots dk_n \end{aligned} \quad (14)$$

for $f \in \mathcal{S}_{n,j}$. We note that the norms $\|\cdot\|_{1,d-1}$ on \mathcal{S}_n in the proof of the temperedness of \hat{G}_n in Subsection 4.2 of [3] can be replaced by norms $\|\cdot\|'_{1,d+1}$ with $\|g\|'_{K,L}$, $K, L \in \mathbb{N}_0$, defined as

$$\sup_{\substack{k_1, \dots, k_n \in \mathbb{R}^d \\ 0 \leq |\beta_1|, \dots, |\beta_n| \leq K}} \left| \prod_{l=j}^{j+1} (1 + |k_l^0|^2)^{-\frac{1}{2}} \left[\prod_{l=1}^n (1 + |\vec{k}_l|^2)^{\frac{L}{2}} \frac{\partial^{|\beta_l|}}{(\partial k_l)^{\beta_l}} \right] g(k_1, \dots, k_n) \right|$$

$\forall g \in \mathcal{S}_n$ without changing the rest of the proof. To see this, it is sufficient to check the simple estimate

$$(1 + |k_r^0|^2)^{1/2} \leq c \prod_{l=1}^n (1 + |\vec{k}_l|^2)^{1/2} \quad \text{for } (k_1, \dots, k_n) \in \text{supp } \hat{G}_n$$

where $r = 1, \dots, n$ and c depends on m_κ , $\kappa = 1, \dots, N$.

Thus, \hat{G}_n is continuous w.r.t. $\|\cdot\|'_{1,d+1}$ (the argument in [3] is formulated only for a special case, but it carries over to the general case by a simple adaptation of notations, cf. [10]).

From the definition of φ_ϵ we get that $\hat{\varphi}_\epsilon(x) = \hat{\varphi}(\epsilon x)$ and $\hat{\varphi}(0) = 1/(2\pi)^{1/2}$. From these properties we get that the product of the two $\hat{\varphi}_\epsilon$ and f in (14) converges to $f(k_1, \dots, k_{j-1}, \vec{k}_j, \vec{k}_{j+1}, k_{j+2}, \dots, k_n)/(2\pi)$ w.r.t the topology induced by $\|\cdot\|'_{1,d+1}$ and thus the limit in (14) exists by the continuity of \hat{G}_n w.r.t. this norm. Furthermore, since $f \rightarrow \|f\|'_{1,d+1}$ defines a Schwartz norm on $\mathcal{S}_{n,j}$, the limit in (14) defines a tempered distribution in $\mathcal{S}'_{n,j}$. ■

We now show the locality of the structure functions for the case of Bosonic locality ($\sigma^{\kappa, \kappa'} = 1$) by application of Theorem 2.1.

Theorem 3.3 *The structure functions G_n , $n \geq 3$, are local.*

Proof. By Theorem 2.1 it suffices to show that $\hat{G}_{n,[j]}(0, 0) = 0$, i.e.

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \hat{G}_n(k_1, \dots, k_{j-1}, [k_j, k_{j+1}], k_{j+2}, \dots, k_n) dk_j^0 dk_{j+1}^0 = 0, \quad (15)$$

where the double integral exists as a distribution in $\mathcal{S}_{n,j}$ as a consequence of Lemma 3.2.

For $j = 1, \dots, n-1$, the commutator in momentum space, $\hat{G}_{n,[j]}$, is given by the following formula:

$$\hat{G}_n(k_1, \dots, k_{j-1}, [k_j, k_{j+1}], k_{j+2}, \dots, k_n) = \left\{ \prod_{l=1}^{j-1} \delta_m^-(k_l) \right.$$

$$\begin{aligned}
& \times \sum_{\kappa_j, \kappa_{j+1}=1}^N \left[\frac{\delta_{m_{\kappa_j}}^-(k_j)}{k_{j+1}^2 - m_{\kappa_{j+1}}^2} + \frac{\delta_{m_{\kappa_{j+1}}}^+(k_{j+1})}{k_j^2 - m_{\kappa_j}^2} - \frac{\delta_{m_{\kappa_j}}^-(k_{j+1})}{k_j^2 - m_{\kappa_{j+1}}^2} \right. \\
& \left. - \frac{\delta_{m_{\kappa_{j+1}}}^+(k_j)}{k_{j+1}^2 - m_{\kappa_j}^2} \right] \prod_{l=j+2}^n \delta_{\vec{m}}^+(k_l) \Big\} \delta\left(\sum_{l=1}^n k_l\right) \quad (16)
\end{aligned}$$

Here we have used the notation $\delta_{\vec{m}}^\pm = \sum_{\kappa_l=1}^N \delta_{m_{\kappa_l}}^\pm$. Changing the order of summation in (16), we can replace the expression in the brackets [...] by

$$\left[\frac{\delta_{m_{\kappa_j}}^-(k_j)}{k_{j+1}^2 - m_{\kappa_{j+1}}^2} + \frac{\delta_{m_{\kappa_{j+1}}}^+(k_{j+1})}{k_j^2 - m_{\kappa_j}^2} - \frac{\delta_{m_{\kappa_{j+1}}}^-(k_{j+1})}{k_j^2 - m_{\kappa_j}^2} - \frac{\delta_{m_{\kappa_j}}^+(k_j)}{k_{j+1}^2 - m_{\kappa_{j+1}}^2} \right]$$

We can now evaluate the integrals in (15) in the following way: First the delta distributions $\delta_{m_{\kappa_j}}^\pm(k_j)$ ($\delta_{m_{\kappa_{j+1}}}^\pm(k_{j+1})$) are being used to evaluate out the integration over k_j^0 (over k_{j+1}^0). Then, we use the delta distribution $\delta(\sum_{l=1}^{j-1} k_l^0 \pm \omega_{j, \kappa_j} + k_{j+1}^0 + \sum_{l=j+2}^n k_l^0)$ ($\delta(\sum_{l=1}^{j-1} k_l^0 + k_j^0 \pm \omega_{j+1, \kappa_{j+1}} + \sum_{l=j+2}^n k_l^0)$) to evaluate the integral over k_{j+1}^0 (k_j^0 , respectively) where $\omega_{l, \kappa_l} = \sqrt{|\vec{k}_l|^2 + m_{\kappa_l}^2}$. As the result we get

$$\left\{ \prod_{l=1}^{j-1} \delta_{\vec{m}}^-(k_l) \sum_{\kappa_j, \kappa_{j+1}=1}^N \left[\dots \right] \prod_{l=j+2}^n \delta_{\vec{m}}^+(k_l) \right\} \delta\left(\sum_{l=1}^n \vec{k}_l\right)$$

with the expression [...] given by

$$\begin{aligned}
& \left[\frac{1}{2\omega_{j+1, \kappa_{j+1}} \left((\omega_{j+1, \kappa_{j+1}} + a)^2 - \omega_{j, \kappa_j}^2 \right)} \right. \\
& \left. + \frac{1}{2\omega_{j, \kappa_j} \left((\omega_{j, \kappa_j} - a)^2 - \omega_{j+1, \kappa_{j+1}}^2 \right)} - \overleftrightarrow{j, j+1} \right],
\end{aligned}$$

with $a = \sum_{l=1, l \neq j, j+1}^n k_l^0$. Here the notation $\overleftrightarrow{j, j+1}$ symbolizes, that the terms standing before the arrow are being repeated with j replaced by $j+1$ and vice versa.

We let $x = \omega_{j, \kappa_j}$, $y = \omega_{j+1, \kappa_{j+1}}$ and we want to show that

$$\frac{1}{2y((y+a)^2 - x^2)} + \frac{1}{2x((x-a)^2 - y^2)} - \overleftrightarrow{x, y} = 0 \quad (17)$$

for $x, y, a \in \mathbb{R}$. We get for the left hand side of (17):

$$\begin{aligned}
& \frac{1}{2y(y+x+a)(y-x+a)} + \frac{1}{2x(x+y-a)(x-y-a)} - \xleftrightarrow{x,y} \\
&= \frac{1}{2xy(y-x+a)} \left(\frac{x}{x+y+a} - \frac{y}{x+y-a} \right) - \xleftrightarrow{x,y} \\
&= \frac{1}{2xy(y-x+a)} \left(\frac{x(x-a) - y(y+a)}{(x+y)^2 - a^2} \right) - \xleftrightarrow{x,y}.
\end{aligned}$$

Since $1/(2xy((x+y)^2 - a^2))$ is symmetric in x and y , it remains to show that

$$\frac{x(x-a) - y(y+a)}{y-x+a} - \xleftrightarrow{x,y} = 0.$$

This is equivalent to

$$(x-y+a) [x^2 - xa - y^2 - ya] - \xleftrightarrow{x,y} = 0.$$

Carrying out the multiplication we get

$$\begin{aligned}
& (\underbrace{x^3}_1 - \underbrace{x^2a}_2 - \underbrace{xy^2}_3 - \underbrace{xya}_4) + (-\underbrace{yx^2}_3 + \underbrace{xya}_4 + \underbrace{y^3}_1 + \underbrace{y^2a}_5) \\
& + (\underbrace{x^2a}_2 - \underbrace{xa^2}_6 - \underbrace{y^2a}_5 - \underbrace{ya^2}_6) - \xleftrightarrow{x,y} = 0,
\end{aligned}$$

where we have labeled the terms which cancel each other or which together give a expression symmetric in x and y with the numbers 1 to 6. Thus, the above equation holds and the proof is finished. ■

Remark 3.4 (i) In the definition of the structure functions we can multiply the distributions $\delta_{m_\kappa}^\pm$ and $1/(k^2 - m_\kappa^2)$ by a weight $\lambda_\kappa \in \mathbb{C}$ and we obtain the locality of the “weighted” structure functions by the same arguments as used in the proof of Theorem 3.3. By an approximation of the integral by Riemannian sums and making use of the fact that the limit of local distributions is again local, one immediately obtains the locality of the distributions

$$\begin{aligned}
& \int_0^\infty \cdots \int_0^\infty \left\{ \sum_{j=1}^n \prod_{l=1}^{j-1} \delta_{m_l}^-(k_l) \frac{1}{k_j^2 - m_j^2} \right. \\
& \times \left. \prod_{l=j+1}^n \delta_{m_l}^+(k_l) \right\} \delta\left(\sum_{l=1}^n k_l\right) \rho(dm_1^2) \cdots \rho(dm_n^2) \quad (18)
\end{aligned}$$

for a (sufficiently regular) locally finite and polynomially bounded complex measure ρ and, in a second step of approximation, even for more irregular distributions ρ . This re-establishes (and even slightly generalises) the result of Theorem 7.10 of [2].

(ii) In the proof of the locality of the two point function of the free field and in Theorem 3.3 the polynomial in \vec{q}_- was the simplest polynomial, i.e. the zero function. This made it particularly simple to apply the criterion Theorem 2.1. One can expect an analogous result for all those Wightman functions in momentum space, where the value of the Wightman function falls to zero whenever the difference of two subsequently following momenta gets very large. Such a behaviour can be justified in a number of physical situations where the impact due to interaction declines if the difference of momenta gets very large.

4 Theorem 2.1 and the Jost-Lehmann-Dyson representation

In this section we briefly compare the characterisation of locality in momentum space given in Theorem 2.1 with the integral representation of causal commutators by Jost, Lehmann and Dyson (JLD): In [6, 9] JLD consider matrix elements of the form

$$f_{\Psi_1, \Psi_2}(\xi_-) = i \langle \Psi_1, [\phi(-\xi_-/2), \phi(\xi_-/2)] \Psi_2 \rangle \quad (19)$$

where Ψ_1, Ψ_2 are vectors¹ from the standard domain \mathcal{D} in the representation Hilbert space \mathcal{H} of the Wightman quantum field (see [11]) $\phi(x)$. For simplicity we only consider the case when $\phi(x)$ is a Bosonic, Hermitean and scalar field. Also, we assume that the above expression exists as a distribution in ξ_- which is essentially equivalent with the assumption that $\langle \Psi_1, [\phi(x), \phi(y)] \Psi_2 \rangle$ is a function in $\xi_+ = (x+y)/2$ s.t. one can set $\xi_+ = 0$. Furthermore, [6, 9] require the decomposability of $f_{\Psi_1, \Psi_2}(\xi_-)$ into an advanced and retarded part. This is essentially the weak time zero field condition 1.1. We do not want to enter into technical details and we assume that $f_{\Psi_1, \Psi_2}(\xi_-)$ is a function in $\xi_-^0 \ \forall \Psi_1, \Psi_2 \in \mathcal{D}$ (in the sense of Cond 1.1) when smeared out in $\vec{\xi}_-$. Let

¹In [6, 9] these vectors are chosen to be (improper) eigen vectors of the energy-momentum operator P , however this is not important in the present context.

$\hat{f}_{\Psi_1, \Psi_2}(q_-)$ be the Fourier-transform of $f_{\Psi_1, \Psi_2}(\xi_-)$. Then, Theorem 2.1 takes the following form:

Corollary 4.1 *Let the above assumptions be fulfilled. Then $\phi(x)$ is local if and only if $\int_{\mathbb{R}} \hat{f}_{\Psi_1, \Psi_2}(q_-) dq_-^0$ is a polynomial in \vec{q}_- $\forall \Psi_1, \Psi_2 \in \mathcal{D}$.*

In Eq. (12) of [6] the JLD-representation is given for the Fourier transform of the matrix element (19) of a local quantum field $\phi(x)$:

$$\begin{aligned} \hat{f}_{\Psi_1, \Psi_2}(q_-) &= \int_{\mathbb{R}^3} \int_0^\infty \varepsilon(q_-^0) \delta((q_-^0)^2 - (\vec{q}_- - \vec{u})^2 - \kappa^2) \\ &\times [\Phi_1(\vec{u}, \kappa^2) + q_-^0 \Phi_2(\vec{u}, \kappa^2)] d\kappa^2 d\vec{u} \end{aligned} \quad (20)$$

where $\varepsilon(q_-^0) = \text{sign}(q_-^0)$ and Φ_1, Φ_2 are uniquely determined (generalised) functions with support properties depending on the spectrum of Ψ_1, Ψ_2 . We furthermore assume that Φ_1 and Φ_2 are sufficiently integrable in order to make shure that the above representation exists and that $f_{\Psi_1, \Psi_2}(\xi_-)$ fulfils the weak time zero field condition. We then get by straight forward calculations:

$$\int_{\mathbb{R}} \hat{f}_{\Psi_1, \Psi_2}(q_-) dq_-^0 = \int_{\mathbb{R}^3} \int_0^\infty \Phi_2(\vec{u}, \kappa^2) d\kappa^2 d\vec{u} = C, \quad (21)$$

where C is a constant and $C = 0$ for an antisymmetric Bosonic commutator. We thus see that locality of the JLD-representation (20) is described by Corollary 4.1 with the additional restriction that all polynomials in \vec{q}_- are zero. This can be considered to be physically sufficient, cf. Remark 3.4 (ii), but it does not exhaust all mathematical cases, as we shall explain using the structure functions G_n , $n \geq 4$, of Section 3 for the simplest case where $N = 1$ and $m_\kappa = m > 0$:

Let $M_n = M_n(k_1, \dots, k_n)$ be a fully Lorentz invariant, symmetric (under exchange of arguments) and real polynomial. It is easy to verify that $\hat{W}_n^T = M_n \cdot \hat{G}_n$ also is the Fourier transform of a local Wightman distribution, cf. [1]. It is also easy to modify Lemma 3.2 and to verify the weak time zero field condition for all such \hat{W}_n^T . It can be shown by explicit, but lengthy calculations that the right hand side of Eq. (15) with \hat{G}_n replaced by \hat{W}_n^T can not be equal to zero for some M_n with sufficiently high degree in any of the variables k_l (≥ 4 is required at least).

Avoiding such tiresome calculations we give the following abstract argument: Suppose that there is a JLD-representation (with properties specified

above) for each such \hat{W}_n^T . By Eq. (21) the right hand side of Eq. (15) with \hat{G}_n replaced by $\hat{W}_{n,[j]}^T$ would be zero for all M_n . Furthermore, fixing momenta $k_1, \dots, k_{j-1}, \vec{k}_j, \vec{k}_{j+1}, k_{j+1}, \dots, k_n$ to compact sets, we see that also k_j^0, k_{j+1}^0 only run over compact sets. Thus we can approximate in Eq. (15) a C^∞ but non-analytic function \tilde{M}_n by polynomials M_n on such compact sets. Taking the limit, we see that also the r.h.s. of Eq. (15) would vanish for \hat{G}_n replaced with $\hat{W}_n^{T\infty} = \tilde{M}_n \cdot \hat{G}_n$. Then, by Theorem 2.1, $\hat{W}_n^{T\infty}$ would be local and by [1, Theorem 4.5] \tilde{M}_n taken on-shell would be the (truncated) scattering amplitude of a local relativistic quantum field theory (with indefinite metric). This, however, is in contradiction with crossing-symmetry, see e.g. [7] where analyticity on certain on-shell regions has been proved². Thus, the JLD-representation can not hold for all causal commutators of \hat{W}_n^T with arbitrary polynomial multiplier M_n .

The reason why the JLD-representation tacitly rules out some cases of causal commutators is the following: In [6] p. 1461 it is required that the distributional product $f(\xi_-)\delta(\xi_-^2 - |y|^2)$ exists where $y \in \mathbb{R}^2$. Using Eqs. (18)–(20) of [6] it is easy to see that there are some distributions $f(\xi_-)$ vanishing for $\xi_-^2 < 0$ s.t. the above distributional product does not exist, take e.g. $d = 4$, $f(\xi_-) = \delta'(\xi_-^1)\delta'(\xi_-^2)\delta'(\xi_-^3)$ which is constant in ξ_-^0 and thus fulfils the weak time zero field condition. As we have demonstrated, at least from a mathematical point of view, also such cases should be taken into account in order to obtain a (more) complete characterisation of locality in momentum space.

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²I am grateful to D. Buchholz for pointing out to me that crossing symmetry is being violated by such approximation arguments.

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